Nonlinear Analysis: Modelling and Control, Online First, 1–17
https://doi.org/10.15388/namc.2024.29.35110

Vilnius University Press

Containing an epidemic in the case of running out of treatment: A switched system approach

Shraddha Salwahan[a], **Syed Abbas**[a], **Abdessamad Tridane**[b,1]

[a]School of Mathematical and Statistical Sciences,
Indian Institute of Technology Mandi, 175005, H.P., India

[b]Department of Mathematical Sciences,
United Arab Emirates University,
Al Ain P.O. Box 15551, UAE
a-tridane@uaeu.ac.ae

Received: January 18, 2023 / **Revised:** March 16, 2024 / **Published online:** April 24, 2024

Abstract. In this paper, we discuss an epidemic switched system. A susceptible–infected–treated model is considered. The course of an epidemic is profoundly influenced by the allocation of resources. If these resources are limited, then we need to devise an optimal distribution strategy. One significant case to study is when the drug supply is insufficient. We study a control problem that minimizes the total outbreak size of the epidemic and optimizes the rate of vaccination/isolation control by minimizing the suitable functional subject to resource constraints. In the end, simulations are performed for illustrations.

Keywords: switched system, epidemic model, optimal control.

1 Introduction

Hybrid systems are defined as dynamical systems whose state has two components. The first one evolves in a continuous set, and the other evolves in a discrete one. The evolution is either through a differential equation, difference equation or according to some transition logic-based rule. These systems are used to model a variety of physical, biological, and engineering systems that exhibit this type of behavior, such as aircraft control systems, automotive systems, and biological systems. For the literature on hybrid systems, [20, 21] can be referred.

Switched systems are a type of hybrid system that occurs in many real-life situations and involves switching between many subsystems based on a variety of conditions. A switched system is made up of a family of continuous-time subsystems and a rule that governs the switching between them. Switched systems are hybrid systems that imitate a wide range of real-world complex systems, including mechanical systems, the

[1]The author is supported by UPAR grant No. 125S12.

automobile industry, air traffic control, robotics, integrated circuit design, multimedia, manufacturing, and chemical processes. Some of the applications of switched systems in infectious disease modeling are given in the papers [17, 18]. This article focuses on applying such a system for disease modeling.

One of the most significant issues in the modern period is the impact of infectious diseases on society. Infectious illness outbreaks result in thousands of deaths and significant expenditures to contain the sickness. Efforts such as sanitation, immunization, and treatment are made to eliminate these diseases. In literature, it can be seen that there is a long and distinguished history of studying infectious diseases with the help of mathematical modeling, for example, [13, 14]. As resources are limited, the important question is to find the optimal way to use the available resources. Optimal control theory is applied to mathematical models with one or many controls, such as vaccination and isolation, to eradicate the disease. Some of the earlier works in this area were by Abakuks in 1972 [1–3]. These papers studied the optimal control of vaccination and isolation strategies for a simple SIR epidemic model. The cost was taken as proportional to the number of infectives and isolated individuals. Whereas in 1975, in [26], the continuous epidemic paths were allowed to run their course until termination, and rates were subject to control. The restriction on the number of isolated and vaccinated was used by removing the unrealistic assumption that, at any given time, an arbitrary number of individuals can be isolated and vaccinated. The observation about [1,26] is that the optimal control should be applied at the beginning of an outbreak or not at all. After these papers, there has been much research on the control of epidemics and infectious diseases in the past few decades, for instance, [12, 24, 28]. A lot of work has been done by epidemiologists to control COVID-19 as well [6, 8]. In all these works, there was no assumption of the possibility of shortage or running out of control measures. By adding a switch to an infectious disease model, we can have different results depending on the model's specific context and assumptions. In general, a switch can represent a change in population or environmental behavior that affects disease spread. Including a switch can help to explain observed phenomena that a basic model cannot explain, for example, only by adding a switch can we explain the case of a drug runout.

Resource allocation plays a vital role in the control of an epidemic [7, 10]. Since the resources are often limited, in preparation for an outbreak, it is essential to determine the optimal distribution of limited resources to control the epidemic. These resources could be either a fixed amount of vaccine or other stockpiled drugs [4, 15] or/and the limited number of facilities and beds for isolation and quarantine [12]. However, in situations where vaccination supplies run out, and treatment options are limited, it is clear that we need another optimal distribution model for the control resources available. Our objective is to answer the following question via a simple modified SIR model: In the case of limited control resources available during an epidemic and the essential treatment drug that could run out, what is the best control strategy for minimizing the total epidemic size? In this situation, we must investigate optimal control for switched systems. The number of switches, their order, and the switching times are all factors that must be taken into consideration when solving a general optimal control problem for switched systems. The optimal control problem of switched systems is studied in [27, 29]. The problem

of optimal control of a switched system with predefined order is investigated in [23]. Optimal control of switched systems can have wide applications to infectious disease modeling.

The main motivation for this paper is to minimize the number of infected and optimize the isolation and vaccination rates. An extension of the SIR model discussed in [9] is considered. We start with the calculation of the reproduction number and the final-size relation. Compared to the papers mentioned earlier, this work's main contribution is that a switched system is considered for the possibility of the drug running out. The switch is considered in three different scenarios, i.e., when the drugs are enough for the whole duration of the epidemic, when drugs run out before the peak in the number of infections, and when the drugs run out after the peak in the number of infections. The optimal control theory for switched systems helps in resource allocation even if the drugs run out, by which we can control the epidemic. It is considered that when the drug treatment is available, the reproduction number $\mathcal{R}_0 < 1$, but when there is drug run out, the reproduction number increases to $\mathcal{R}_0 > 1$, which means the disease cannot be controlled. By optimal control theory, the epidemic is controlled when interventions like isolation and vaccination are introduced. We can find research on switched system applications to disease modeling or to optimal control of a disease in the literature, but not both, although it can potentially help us to understand diseases and their control.

This paper is organized as follows. Section 2 defines an epidemic model and uses the next-generation method for reproduction number. Section 3 discusses the final size relation and how it relates to the drugs available at the beginning of the epidemic. We introduce a switch and define the controls for the optimal control problem (OCP) in Section 4. The OCP is solved using the Pontryagin maximum principle in Section 5. Some numerical simulations are performed in Section 6. Conclusions are drawn in Section 7.

2 Model description and analysis

We consider the following epidemic model:

$$\begin{aligned} S' &= -\beta S(I + \delta T), \\ I' &= \beta S(I + \delta T) - (\alpha + \gamma)I, \\ T' &= \gamma I - \eta T \end{aligned} \tag{1}$$

with

$$S(0) > 0, \qquad I(0) > 0, \qquad T(0) = 0.$$

We assume that the total population N is constant $N = S_0 + I_0$ with γ as the rate of treatment. The number of cases of disease during the epidemic is $N - S_\infty$ with $S_\infty = \lim_{t\to\infty} S$. If $\gamma > 0$, the fraction of the treated is $\gamma/(\alpha + \gamma)$. By assuming that each treatment consists of a drug dose each day during the mean treatment period of $1/\eta$ days, the quantity of drugs required is

$$U(\gamma) = \frac{\gamma}{\eta(\alpha + \gamma)}[N - S_\infty].$$

We can write the two extreme cases of drug quantity as follows:

$$U(0) = 0, \qquad U(\infty) = \frac{N - S_\infty}{\eta} > 0.$$

$\mathcal{R}_0$ was defined as the spectral radius of the next-generation matrix by [11]. We calculate the reproduction number for our model using the same method. Let us denote f_i as the influx on infected humans ith compartment. The rate at which individuals transfer into the ith compartment by all other means is denoted as v_i^+, and v_i^- is the rate at which individuals transfer out of the compartment i. So a general equation for ith compartment can be written as

$$\frac{\mathrm{d}x_i}{\mathrm{d}x} = f_i - v_i,$$

where $v_i = v_i^+ - v_i^-$. If f and v are the column matrix for f_i and v_i, we get

$$f = \begin{bmatrix} \beta S(I + \delta T) \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} (\alpha + \gamma)I \\ \eta T - \gamma I \end{bmatrix}.$$

Let $\mathcal{F}$ and $\mathcal{V}$ be the jacobian matrix of f and v, respectively, calculated at the disease-free equilibrium, i.e.,

$$\mathcal{F} = \frac{\mathrm{d}f_j}{\mathrm{d}x_i}, \qquad \mathcal{V} = \frac{\mathrm{d}v_j}{\mathrm{d}x_i}.$$

The disease-free equilibrium for Eq. (1) is $E^* = (S^*, I^*, T^*) = (N, 0, 0)$. Thus, we get $\mathcal{F}$ and $\mathcal{V}$ as

$$\mathcal{F} = \begin{bmatrix} \beta N & \beta N\delta \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{V} = \begin{bmatrix} \alpha + \gamma & 0 \\ -\gamma & \eta \end{bmatrix}.$$

The matrix $\mathcal{F}\mathcal{V}^{-1}$ is known as the next-generation matrix,

$$\mathcal{F}\mathcal{V}^{-1} = \begin{bmatrix} \beta N & \beta N\delta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\alpha+\gamma)} & 0 \\ \frac{\gamma}{\eta(\alpha+\gamma)} & \frac{1}{\eta} \end{bmatrix} = \begin{bmatrix} \frac{\beta N}{(\alpha+\gamma)} + \frac{\beta N\delta\gamma}{\eta(\alpha+\gamma)} & \frac{\beta N\delta}{\eta} \\ 0 & 0 \end{bmatrix}.$$

Hence, the reproduction number is

$$\mathcal{R}_0 = \frac{\beta N}{(\alpha + \gamma)}\left(1 + \frac{\delta\gamma}{\eta}\right).$$

Since $\mathcal{R}(\gamma)$ is a decreasing function of γ if $\eta > \delta\alpha$, this means that the treatment is beneficial.

If there is no treatment, i.e., $\gamma = 0$,

$$\mathcal{R}_0^0 = \mathcal{R}(\gamma = 0) = \frac{\beta N}{\alpha}.$$

3 The final size relation

The final size relation is given by [5]

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0(\gamma)\left(1 - \frac{S_\infty}{N}\right).$$

Similar to the results in [13], we have the following lemma.

Lemma 1. *For $0 < S_0 < N$ and a constant K positive, the equation*

$$\ln \frac{S_0}{x} = K\left(1 - \frac{x}{N}\right)$$

has a unique solution $x < N/K$.

Proof. The left side of the equation decreases monotonically from $+\infty$ as $x \to 0+$ to 0 for $x = S_0$. The right side decreases monotonically from R for $x = 0$ to a positive value at $x = S_0$. Thus there is a solution S_∞, $0 < S_\infty < S_0$. Since the second derivative of the left side of the equation is positive, while the second derivative of the right side is zero, this solution is unique. For $x = N/K$, the left side is $\ln(S_0K/N) < \ln K$, while the right side is $K - 1$. Since $\ln K < K - 1$ for $K > 0$, it follows that at $x = N/K$, the left side of the equation is less than the right side, and this implies that $S_\infty < N/K$, which proves the lemma. □

Implicit differentiation of the final size relation with respect to γ gives

$$\left(\frac{1}{S_\infty(\gamma)} - \frac{\mathcal{R}_0(\gamma)}{N}\right) S'_\infty(\gamma) = -\mathcal{R}'_0(\gamma)\left[1 - \frac{S_\infty(\gamma)}{N}\right].$$

The lemma plus the fact that $\mathcal{R}'_0(\gamma) < 0$ imply that $S'_\infty(\gamma) > 0$.

Let the quantity of drugs available at time t be $D(t)$, and let D_0 be the drugs available at the beginning of the epidemic. If the supply of drugs is limited, then we have the following cases:

(i) When $D_0 > U(\infty)$, (1) is applicable for all t.
(ii) When $D_0 < U(\infty)$, there is a value γ_1 such that $D_0 < U(\gamma)$ if $\gamma > \gamma_1$. In this case, if $\gamma_1 < \gamma$, model (1) is applicable only until the drug supply runs out, after which we must take $\gamma = 0$.

Therefore, it is necessary to consider the possibility of drug run out separately from the case in which the model is valid for all t. As the case in which the drug supply is sufficient has been studied completely, we consider only the case of insufficient drug supply. If drugs run out at time τ, then at time τ, we must change γ to zero. Thus we replace the constant γ by the function

$$\gamma(t) = \begin{cases} \gamma, & 0 \leqslant t < \tau, \\ 0, & t \geqslant \tau. \end{cases}$$

The number of members entering treatment at time t is $\gamma I(t)$, which means that the total number of members treated up to time τ is $\int_0^\tau \gamma I(s)\,\mathrm{d}s$, and by integrating the third equation of (1) we see that this is equal to $\int_0^\infty \eta T(s)\,\mathrm{d}s$. Thus,

$$D(t) = D_0 - \int_0^t \gamma I(s)\,\mathrm{d}s,$$

and the supply of drugs runs out at time τ if $\int_0^\tau \gamma I(s)\,\mathrm{d}s = D_0$. In this case, drug treatment must cease at time τ, and then

$$D_0 = \int_0^\tau \gamma I(s)\,\mathrm{d}s = \int_0^\infty \eta T(s)\,\mathrm{d}s.$$

In practice, we would solve system (1) and calculate the supply of drugs needed for a given choice of treatment rate γ. Then a choice of D_0 less than this amount will guarantee that the drug supply will run out.

Division and then integration of the first equation of (1) give

$$\ln \frac{S_0}{S_\infty(\gamma)} = \beta(\hat{I} + \delta\hat{T}).$$

Adding the first two equations of (1) and integrating, we obtain

$$\begin{aligned} N - S_\infty(\gamma) = (\alpha+\gamma)\hat{I} &= \alpha\hat{I} + \int_0^\infty \eta T(t)\,\mathrm{d}t \\ &= \alpha\hat{I} + D_0. \end{aligned}$$

Thus,

$$\beta\hat{I} = \frac{\beta}{\alpha}\big[N - S_\infty(\gamma)\big] - \frac{\beta}{\alpha}D_0.$$

Then

$$\begin{aligned} \ln \frac{S_0}{S_\infty(\gamma)} &= \frac{\beta}{\alpha}[N - S_\infty] - \frac{\beta\eta}{\alpha}\hat{T} + \beta\delta\hat{T} \\ &= \frac{\beta}{\alpha}[N - S_\infty] + \beta\left(\delta - \frac{\eta}{\alpha}\right)\hat{T} \\ &= \mathcal{R}_0^0\left[1 - \frac{S_\infty(\gamma)}{N}\right] + \beta\left(\delta - \frac{\eta}{\alpha}\right)D_0. \end{aligned} \tag{2}$$

Thus, in the case of drug run out, $S_\infty(\gamma)$ is independent of γ. This shows that if the drug supply is insufficient to last the duration of the epidemic with the chosen treatment rate, using up the drug supply does not affect the final size of the epidemic. Note, in addition, that in case 2, while drugs run out for some values of γ, there are larger values of γ for which the drug supply is sufficient and for which the epidemic size is still smaller.

If γ is chosen so that there is no run out of the drug supply, model (1) is valid for all t. Integration of the sum of the first two equations gives

$$N - S_\infty(\gamma) = (\alpha + \gamma)\hat{I}.$$

Integration of the third equation of (1) gives

$$\gamma \hat{I} = \eta \hat{T},$$

and from this combined with (2) we obtain the usual final size relation

$$\ln \frac{S_0}{S_\infty(\gamma)} = \mathcal{R}_0(\gamma)\left[1 - \frac{S_\infty(\gamma)}{N}\right].$$

4 Treatment model with a switch

To study the impact of limited drug supply, as discussed in Section 3, we shall now consider a switch in model (1):

$$\begin{aligned} \dot{S} &= -\beta S(I + \delta T), \\ \dot{I} &= \beta S(I + \delta T) - \big(\alpha + \gamma(t)\big)I, \\ \dot{T} &= \gamma(t)I - \eta T \end{aligned} \tag{3}$$

with γ as a piece-wise continuous function of time defined as

$$\gamma(t) = \begin{cases} \gamma, & 0 \leqslant t < \tau, \\ 0, & t \geqslant \tau. \end{cases} \tag{4}$$

If we call $x = (S, I, T)^{\mathrm{T}}$, we get a switched system

$$\dot{x}(t) = f_s\big(x(t)\big), \quad s = 1, 2,$$

where f_1 and f_2 can be defined as

$$f_1 = \begin{bmatrix} -\beta S(I + \delta T) \\ \beta S(I + \delta T) - (\alpha + \gamma)I \\ \gamma(t)I - \eta T \end{bmatrix}, \qquad f_2 = \begin{bmatrix} -\beta S(I + \delta T) \\ \beta S(I + \delta T) - \alpha I \\ -\eta T \end{bmatrix}.$$

The goal is to study the effect of measures like vaccination and isolation on controlling the epidemic, which is modeled by a switched system. Now, to mimic the real-life scenario, if we consider such measures in the model, we will have to consider them as controls. The affordability of vaccination and isolation depends on the resources and funds of different countries/states.

In this situation, we have considered two control variables: u_v, the rate of vaccination of the susceptible population, and u_i, the rate of isolation of the infected people. Note

that the two controls must have upper and lower bounds. The lower bound of u_v, $u_{v_{\min}}$, represents nonvaccination of the population, and the high bound, $u_{v_{\max}}$, means the maximum capacity of vaccination that can be administrated during an epidemic. Similarly, the lower and higher bounds of u_i, $u_{i_{\min}}$ and $u_{i_{\max}}$, respectively, represent the no isolation capacity, the high possible isolation capacity that can be created during a health crisis.

It is clear that values of the upper bounds depend on the nature of the infectious disease and the health care capacity of the country where the disease is spreading. Some countries have more isolation capacities than others [22], and the vaccination supply can also be facing some shortage as was the case for COVID-19 [25]. In our case, we will assume that $u_{i_{\min}} = u_{v_{\min}} = 0$, which means that lower bounds of our control are the case where no vaccination is administrated, and nobody is isolated. The boundedness of the controls reflects the assumption that there is a maximum rate of isolation and vaccination, respectively, that can be implied during the epidemic as it directly impacts a country's funds. Our objective is to solve our problem by adapting these techniques to our case.

5 Optimal control approach

Consider a switched system with k subsystems

$$\dot{x}(t) = f_r\big(x(t), u(t)\big), \quad r = 1, \dots, k.$$

Let U_H and U_L be the upper and lower bounds of the control $u(t)$. We aim to minimize

$$J(x, u, t) = \int_{t_0}^{t_f} L\big(x(t), u(t), t\big)\,\mathrm{d}x$$

with bounded controls

$$U_L \leqslant u(t) \leqslant U_H.$$

We will construct the Hamiltonian, to solve the above optimal control problem, i.e.,

$$\mathfrak{H}_r(x, p, u, t) = L(x, p, u, t) + p^{\mathrm{T}} f_r(x, u, t),$$

where p^{T} is the costate vector.

Now using the Pontryagin maximum principle (PMP) to find the optimal control:

$$\begin{aligned}
&\dot{x}(t) = \frac{\partial \mathfrak{H}_r(x, p, u, t)}{\partial p},\\
&\dot{p}(t) = \frac{\partial \mathfrak{H}_r(x, p, u, t)}{\partial x} \quad \text{(adjoint equation)},\\
&\begin{cases}
u* = U_L & \text{if } \frac{\partial \mathfrak{H}_r(x,p,u,t)}{\partial u} < 0,\\
U_L \leqslant u* \leqslant U_H & \text{if } \frac{\partial \mathfrak{H}_r(x,p,u,t)}{\partial u} = 0,\\
u* = U_H & \text{if } \frac{\partial \mathfrak{H}_r(x,p,u,t)}{\partial u} > 0
\end{cases} \quad \text{(optimality condition)}.
\end{aligned}$$

Using the above approach, we shall now discuss the optimal control problem for model (3).

5.1 Optimal control problem (OCP) for switched treatment model

Let us fix $u_{v_{\max}}, u_{i_{\max}} \geqslant 0$ and determine the control for model (3)–(4) that minimizes the total outbreak size over the course of the epidemic and the cost of vaccination/isolation control by minimizing

$$J = u_v^2 + u_i^2 + \int_{t_0}^{T} I(t)\,\mathrm{d}t \tag{5}$$

subject to:

- $S(t_0) = S_0,\ I(t_0) = I_0,\ T(t_0) = 0$,
- $(u_i(t_k), u_v(t_r)) \in [0, u_{i_{\max}}] \times [0, u_{v_{\max}}]$ for all $(k, r) \in (\mathbb{N}^*)^2$,
- u_v and u_i are piecewise continuous functions, where $u_v : [0, \infty] \to [0, u_{v_{\max}}]$ and $u_i : [0, \infty] \to [0, u_{i_{\max}}]$.

Our goal is to conclude the effect of the shortage of drug treatment on the management of strategies of control policy. When it is available and efficient, it is clear that the treatment will reduce the burden of minimizing cost functional (5). But it would be beneficial to determine a relationship between the run-out time and the number of resources to allocate that will help to control an epidemic. Notably, we will investigate the effect of all these two controls in the presence of treatment and its absence. This is very important for the decision maker to understand what it takes to stop the spread of infectious diseases. We can write optimal control problem, where the goal is to minimize

$$J = \int_{t_0}^{T} {u_v}^2 + {u_i}^2 + I(t)\,\mathrm{d}t$$

subject to

$$\dot{x}(t) = f_s\big(x(t)\big) + u_v(t)g\big(x(t)\big) + u_i(t)h\big(x(t)\big),$$

where $s = 1, 2$, i.e., we have two subsystems on $[t_0, t_f]$. Define the terms $x(t)$, $g(x(t))$, $h(x(t))$ as

$$\dot{x}(t) = \begin{bmatrix} \dot{S}(t \\ \dot{I}(t) \\ \dot{T}(t) \end{bmatrix}, \qquad g\big(x(t)\big) = \begin{bmatrix} -S(t) \\ 0 \\ 0 \end{bmatrix}, \qquad h\big(x(t)\big) = \begin{bmatrix} 0 \\ -I(t) \\ I(t) \end{bmatrix}.$$

Initial conditions are $I(t_0) = I_0,\ S(t_0) = S_0,\ T(t_0) = 0$. Here the susceptibles, which are vaccinated are out of the system, and the infected, which are isolated, are still in the system, which is reflected in $g(x)$ and $h(x)$.

Theorem 1. *Suppose u_v^* and u_i^* minimize the optimal control problem. Let $\mathbf{x}^*(t) = (S^*(t), I^*(t), T^*(t))$ denote the optimal solution. Then there exists piecewise C^1 vector function $\mathbf{p}_k^* = (p_{k1}, p_{k2}, p_{k3})$ for $k = 1, 2$ such that*:

(i) $\mathfrak{H}_k(\mathbf{x}, u_i, u_v, \mathbf{p}_k) \geqslant \mathfrak{H}_k(\mathbf{x}^*, u_i^*, u_v^*, \mathbf{p}_k^*)$ *for both the controls at each time* t.
(ii) $\dot{\mathbf{p}}_k^* = -\nabla_x \mathfrak{H}_k(\mathbf{x}^*, u_i^*, u_v^*, \mathbf{p}_k^*)$
(iii) $\partial \mathfrak{H}_k(\mathbf{x}^*, u_i^*, u_v^*, \mathbf{p}_k^*)/\partial u_i = 0$, $\partial \mathfrak{H}_k(\mathbf{x}^*, u_i^*, u_v^*, \mathbf{p}_k^*)/\partial u_v = 0$ *at* $u_i = u_i^*$, $u_v = u_v^*$ (*optimality condition*).

Here Hamiltonian $\mathfrak{H}_k(\mathbf{x}, u_v, u_i, \mathbf{p}_k)$ *is defined as above.*

Proof. To begin, we write the subsystems

$$\text{Subsystem 1} \quad \begin{bmatrix} \dot{S} \\ \dot{I} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} -\beta S(I + \delta T) \\ \beta S(I + \delta T) - (\alpha + \gamma(t))I \\ \gamma(t)I - \eta T \end{bmatrix} + u_v \begin{bmatrix} -S \\ 0 \\ 0 \end{bmatrix} + u_i \begin{bmatrix} 0 \\ -I \\ I \end{bmatrix},$$

$$\text{Subsystem 2} \quad \begin{bmatrix} \dot{S} \\ \dot{I} \\ \dot{T} \end{bmatrix} = \begin{bmatrix} -\beta S(I + \delta T) \\ \beta S(I + \delta T) - \alpha I \\ -\eta T \end{bmatrix} + u_v \begin{bmatrix} -S \\ 0 \\ 0 \end{bmatrix} + u_i \begin{bmatrix} 0 \\ -I \\ I \end{bmatrix}.$$

The system switches from subsystem 1 to subsystem 2 at time $t = \tau$. Our aim is to minimize $J(S, I, T, u_v, u_i, t)$,

$$J = \int_{t_0}^{T} {u_v}^2 + {u_i}^2 + I(t)\,\mathrm{d}t$$

with bounded controls

$$0 \leqslant u_v \leqslant u_{v_{\max}}, \qquad 0 \leqslant u_i \leqslant u_{i_{\max}}.$$

Hamiltonians for subsystems are

$$\mathfrak{H}_1 = {u_v}^2 + {u_i}^2 + I(t) + \begin{bmatrix} p_{11} & p_{12} & p_{13} \end{bmatrix} \begin{bmatrix} -\beta S(I + \delta T) - u_v S] \\ \beta S(I + \delta T) - (\alpha + \gamma(t))I - u_i I \\ \gamma(t)I - \eta T + u_i I \end{bmatrix},$$

$$\mathfrak{H}_2 = {u_v}^2 + {u_i}^2 + I(t) + \begin{bmatrix} p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} -\beta S(I + \delta T) - u_v S] \\ \beta S(I + \delta T) - \alpha I - u_i I \\ -\eta T + u_i I \end{bmatrix}.$$

We use the maximum principle to find optimal controls u_v^*, u_i^*. Adjoint equations are

$$\dot{p}_{11}(t) = -\frac{\partial \mathfrak{H}_1}{\partial S}, \qquad \dot{p}_{12}(t) = -\frac{\partial \mathfrak{H}_1}{\partial I}, \qquad \dot{p}_{13}(t) = -\frac{\partial \mathfrak{H}_1}{\partial T},$$

$$\dot{p}_{21}(t) = -\frac{\partial \mathfrak{H}_2}{\partial S}, \qquad \dot{p}_{22}(t) = -\frac{\partial \mathfrak{H}_2}{\partial I}, \qquad \dot{p}_{23}(t) = -\frac{\partial \mathfrak{H}_2}{\partial T},$$

and we get

$$\begin{aligned} \dot{p}_{11} &= -\big[p_{11}(-\beta(I + \delta T) - u_v) + p_{12}\beta(I + \delta T)\big], \\ \dot{p}_{12} &= -\big[p_{11}(-\beta S) + p_{12}(\beta S - \big(\alpha + \gamma(t)\big) - u_i) + p_{13}(\gamma(t) + u_i) + 1\big], \\ \dot{p}_{13} &= -\big[p_{11}(-\beta S\delta) + p_{12}\beta S\delta - p_{13}\eta\big], \end{aligned}$$

$$\dot{p}_{21} = -\big[p_{21}\big(-\beta(I+\delta T) - u_v\big) + p_{22}\beta(I+\delta T)\big],$$
$$\dot{p}_{22} = -\big[p_{21}(-\beta S) + p_{22}(\beta S - \alpha - u_i) + p_{23}u_i + 1\big],$$
$$\dot{p}_{23} = -\big[p_{21}(-\beta S\delta) + p_{22}\beta S\delta - p_{13}\eta\big].$$

From the optimality condition we get the equations

$$u_v = \frac{p_{11}S}{2}, \quad u_i = \frac{(p_{12}-p_{13})I}{2} \quad \text{for } t \leqslant \tau,$$
$$u_v = \frac{p_{21}S}{2}, \quad u_i = \frac{(p_{22}-p_{23})I}{2} \quad \text{for } t \geqslant \tau. \qquad \square$$

6 Numerical simulations

For numerical results, we use the forward-backward sweep algorithm (Algorithm 1). A general idea of the algorithm is given below. The vector approximation for state and adjoint are given as $x = (x_1, \ldots, x_{N+1})$ and $\lambda = (\lambda_1, \ldots, \lambda_{N+1})$.

For numerical results, the values of parameters are motivated from [19]. We assume that $\alpha = 0.69$, $\beta = 0.002$, $\gamma = 0.4$, $\delta = 0.009$, $\eta = 0.9$. The total population is taken $N = 500$ units. The basic reproduction number for each case is $\mathcal{R}_0 = 0.921$ and $\mathcal{R}_0^0 = 1.45$.

In Fig. 1, the peak in infections without the effect of control is approximately 190 units around 100 days. Then we see the effect of both the controls on it (blue line). Figures 3 and 5, respectively, illustrate the effect of controls in the presence of switch before and after the peak in the number of infections. Moreover, for the numerical example above, the controls are bounded as they are the rates of vaccination (u_v) and isolation (u_i), which are finite. Bounds are taken as

$$0 < u_i < 0.3, \qquad 0 < u_v < 0.99.$$

In Fig. 1, we see that controls u_i and u_v help decrease the number of susceptibles and infected drastically. The susceptibles decrease and stabilize at around 150 units. Since the number of infected decreases, it implies that the number of treated will also decrease, which is reflected in Fig. 1 for the treated (T) population. From Fig. 2 we see that vaccination control (u_v) is applied to its full potential around 38 days and then decreased, whereas the isolation control (u_i) is applied till 90 days to contain the epidemic.

Algorithm 1. Forward-backward sweep algorithm [16].

Step 1. Start with making an initial estimation for u_i and u_v over the interval of time.

Step 2. Use the initial condition $x(t_0)$, which is denoted by x_1, and the values estimated of u_i and u_v in solving forward in time for x according to its differential equation in the optimality system.

Step 3. Use $\lambda_{N+1} = \lambda(t_1) = 0$, which is the transversality condition along with the values of u_i, u_v, and x and solve backward in time for λ according to its differential equation in the optimality system.

Step 4. Update u_i and u_v by entering the new x and λ values into the characterization of the optimal control.

Step 5. Test the convergence. If the value of each variable in this iteration and the last iteration are almost the same, use the current values as the solutions. If the values are far apart, go back to Step 2.

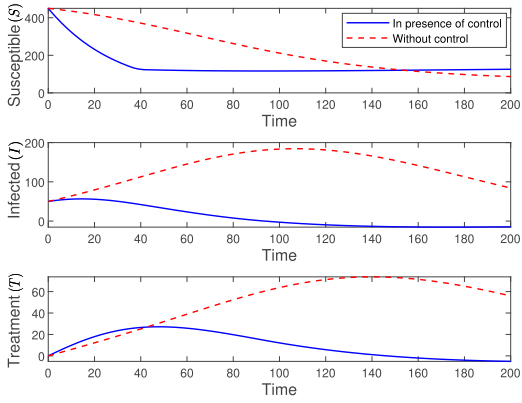

Figure 1. The basic model with and without control.

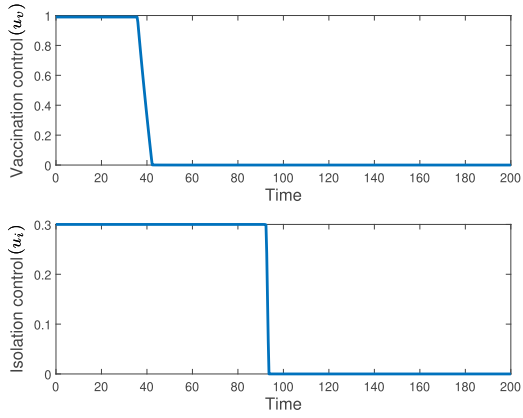

Figure 2. Controls u_v and u_i, i.e., vaccinated and isolated.

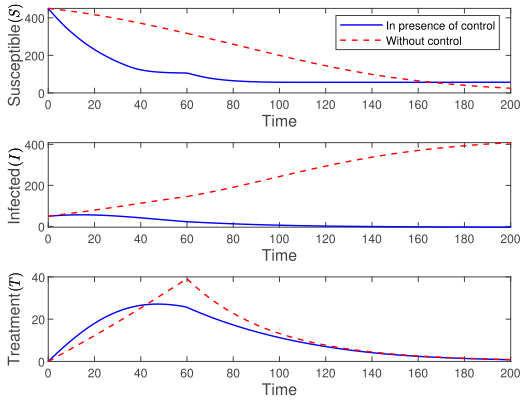

Figure 3. When the switch is introduced at day 60.

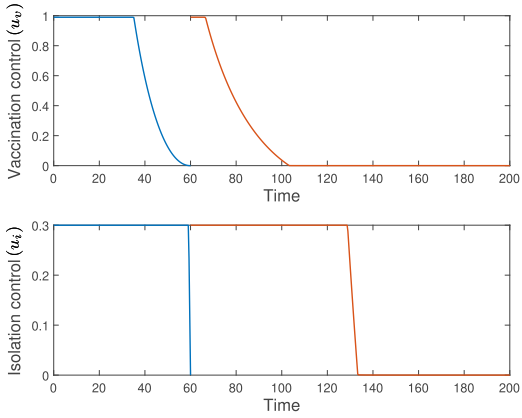

Figure 4. Controls with the switch.

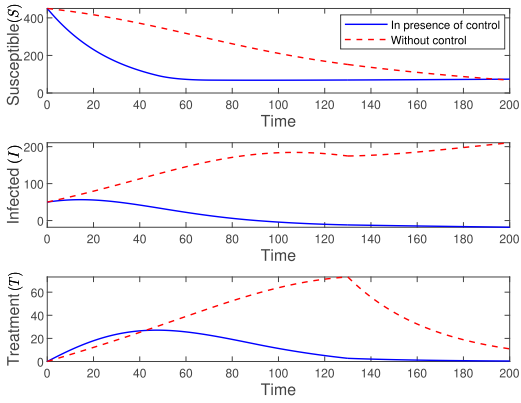

Figure 5. When the switch is introduced at day 130.

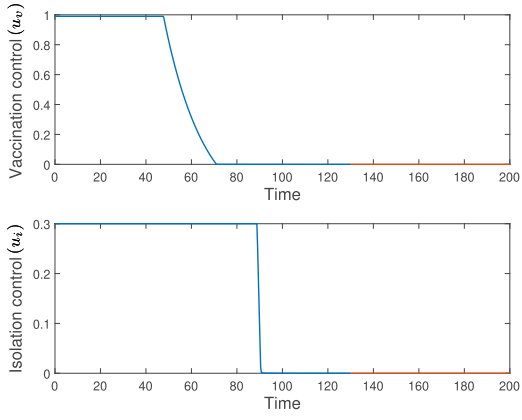

Figure 6. The effect on controls when the switch is introduced after the peak in infection (day 130).

In Fig. 3, the switch is introduced before the peak in infections on day 60. There is no treatment available after 60th day. We see the role of isolation and vaccination control in such conditions. The red (dashed) line shows the situation without control. The number of infected individuals cannot be controlled, and the number of treated individuals decreases immediately. The blue line shows the presence of controls and gives us hope that even though there is no treatment, the epidemic can still be controlled. From Fig. 4 we infer that both the controls need to be on a complete potential before and after the switch vaccinated control is at a high level and decreases gradually to zero around 100 days. The isolation control should be at full capacity till 130th day.

The scenario is entirely different when the switch is introduced after the peak in infections in Fig. 5 on day 130. The number of infections increases instead of decreasing in the absence of control (red dashed line). Since the susceptibles have already reduced significantly, it remains unaffected after the switch in case of control or no control. The number of treated individuals decreases sharply after the switch when there is no control, whereas in the presence of controls, the number of infected individuals decreases. As a result, the number of treated individuals drops, so after the switch, the decline is not sharp.

Here basic reproduction number is 1.44 when there is no treatment, and the reproduction number in the presence of treatment is 0.921. In general, the epidemic can be controlled if $\mathcal{R}_0 < 1$, but we try to contain the epidemic when $\mathcal{R}_0 > 1$ in the presence of control. In Figs. 3 and 5, it is visible that if controls are applied appropriately even after the switch, i.e., when there are drugs, we can still control the epidemic. It is beneficial that the peak in infections is in the presence of treatment. In other words, we can say that the switch should be delayed until after the peak in infections. A delayed switch will ensure that the epidemic is controlled and there is early relaxation in both controls.

7 Conclusion

As the world is facing many waves of different diseases (such as COVID-19), there is a need to be prepared to contain any epidemic or pandemic. Its main aspect is to have access to adequate resources that allow the public health authorities to control the dynamic of the disease spread. However, we witness that access to resources, such as drug treatment, isolation facilities, and vaccination doses, might be challenging or even impossible in many countries. Therefore, we need to investigate how running out of resources can be manageable in the case of a disease outbreak.

In this work, we attempt to study one aspect of this problem by focusing on the case of run out treatment and how we can control the isolation of the population and vaccinate them. First, we find the quantity of drugs required as a function of the number of disease cases during the epidemic. Then we calculate the reproduction number and the final size relation. We adopt the method given in [9]. Furthermore, we prove a lemma for the uniqueness of S_∞ under the given conditions on $\mathcal{R}_0$ and S_0. A term for the quantity of drugs available is defined, i.e., $D(t)$. Now, we calculate the final size relation in terms of $S_\infty(\gamma)$, reproduction number without treatment, and D_0. In the next section,

we considered a susceptible–infected–treated model with two controls, u_i and u_v, and proved its existence. Our work only looked at one aspect of resource limitation in the case of an epidemic (the treatment). However, the problem can also be studied in the case of running out of vaccination, as is the case in many developing countries during the current pandemic. The effort should focus mainly on other control approaches that have been used, such as face masks, testing, and quarantine.

Author contributions. All authors contributed equally.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors would like to thank the associate editor and the anonymous reviewers for their valuable comments and suggestions, which significantly enhanced the quality of this manuscript.

References

1. A. Abakuks, *Some Optimal Isolation and Immunisation Policies for Epidemics*, PhD thesis, University of Sussex, 1972, https://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.446630.
2. A. Abakuks, An optimal isolation policy for an epidemic, *J. Appl. Probab.*, **10**(2):247–262, 1973, https://doi.org/10.2307/3212343.
3. A. Abakuks, Optimal immunisation policies for epidemics, *Adv. Appl. Probab.*, **6**(3):494–511, 1974, https://doi.org/10.2307/1426230.
4. N. Arinaminpathy, J. Savulescu, A.R. Mclean, Effective use of a limited antiviral stockpile for pandemic influenza, *J. Bioethic. Inq.*, **6**(2):171–179, 2009, https://doi.org/10.1007/s11673-009-9164-3.
5. J. Arino, F. Brauer, P. van den Driessche, J. Watmough, J. Wu, A final size relation for epidemic models, *Math. Biosci. Eng.*, **4**(2):159, 2007, https://doi.org/10.3934/mbe.2007.4.159.
6. L. Benahmadi, M. Lhous, A. Tridane, Mathematical modeling of COVID-19 in Morocco and the impact of controlling measures, *Commun. Math. Biol. Neurosci.*, **2021**, 2021, https://doi.org/10.28919/cmbn/5697.
7. J.M. Calabrese, J. Demers, How optimal allocation of limited testing capacity changes epidemic dynamics, *J. Theor. Biol.*, **538**:111017, 2022, https://doi.org/10.1016/j.jtbi.2022.111017.
8. S.S. Chaharborj, S.S. Chaharborj, J.H. Asl, P.S. Phang, Controlling of pandemic COVID-19 using optimal control theory, *Results Phys.*, **26**:104311, 2021, https://doi.org/10.1016/j.rinp.2021.104311.
9. G. Chowell, F. Brauer, The basic reproduction number of infectious diseases: Computation and estimation using compartmental epidemic models, in G. Chowell (Ed.), *Mathematical and Statistical Estimation Approaches in Epidemiology*, Springer, Dordrecht, 2009, pp. 1–30, https://doi.org/10.1007/978-90-481-2313-1_1.

10. C.E. Dangerfield, M. Vyska, C.A. Gilligan, Resource allocation for epidemic control across multiple sub-populations, *Bull. Math. Biol.*, **81**(6):1731–1759, 2019, https://doi.org/10.1007/s11538-019-00584-2.

11. O. Diekmann, J.A.P. Heesterbeek, J.A. Metz, On the definition and the computation of the basic reproduction ratio r 0 in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, **28**(4):365–382, 1990, https://doi.org/10.1007/BF00178324.

12. E. Hansen, T. Day, Optimal control of epidemics with limited resources, *J. Math. Biol.*, **62**(3): 423–451, 2011, https://doi.org/10.1007/s00285-010-0341-0.

13. H.W. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42**(4):599–653, 2000, https://doi.org/10.1137/S0036144500371907.

14. M.J. Keeling, P. Rohani, *Modeling Infectious Diseases in Humans and Animals*, Princeton Univ. Press, Princeton, 2011, https://doi.org/10.1515/9781400841035.

15. S. Kim, Y.B. Seo, J. Lee, Y.S. Kim, E. Jung, Estimation of optimal antiviral stockpile for a novel influenza pandemic, *J. Infect. Public Health*, **15**(7):720–725, 2022, https://doi.org/10.1016/j.jiph.2022.05.012.

16. S. Lenhart, J.T. Workman, *Optimal Control Applied to Biological Models*, Chapman & Hall/CRC, 2007, https://doi.org/10.1201/9781420011418.

17. X. Liu, P. Stechlinski, Infectious disease models with time-varying parameters and general nonlinear incidence rate, *Appl. Math. Modelling*, **36**(5):1974–1994, 2012, https://doi.org/10.1016/j.apm.2011.08.019.

18. X. Liu, P. Stechlinski, *Infectious Disease Modeling: A Hybrid System Approach*, Springer, Cham, 2017, https://doi.org/10.1007/978-3-319-53208-0.

19. N.K. Sacrifice, E. Okyere, N.K. Frempong, S.O. Akindeinde, J.D.G. Ankamah, J.K. Agyen, D. Adedia, An SITR analysis of treatment model of hepatitis b epidemic, *Mathematical Theory and Modeling*, **5**(13):120132, 2015.

20. A.V. Savkin, R.J. Evans, *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*, Springer, New York, 2002, https://doi.org/10.1007/978-1-4612-0107-6.

21. A. Schaft, H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, Lect. Notes Control Inf. Sci., Vol. 251, Springer, London, 2000, https://doi.org/https://doi.org/10.1007/BFb0109998.

22. B. Sen-Crowe, M. Sutherland, M. McKenney, A. Elkbuli, A closer look into global hospital beds capacity and resource shortages during the COVID-19 pandemic, *J. Surg. Res.*, **260**:56–63, 2021, https://doi.org/10.1016/j.jss.2020.11.062.

23. E. Sigal, J.Z. Ben-Asher, Optimal control for switched systems with pre-defined order and switch-dependent dynamics, *J. Optimiz. Theory Appl.*, **161**:582–591, 2014, https://doi.org/10.1007/s10957-013-0411-8.

24. E. Verriest, F. Delmotte, M. Egerstedt, Control of epidemics by vaccination, in *Proceedings of the 2005 American Control Conference, Portland, OR, USA, 2005*, IEEE, 2005, pp. 985–990, https://doi.org/ACC.2005.1470088.

25. O.J. Watson, G. Barnsley, J. Toor, A.B. Hogan, P. Winskill, A.C. Ghani, Global impact of the first year of COVID-19 vaccination: A mathematical modelling study, *Lancet Infect. Dis.*, **22**(9):1293–1302, 2022, https://doi.org/10.1016/S1473-3099(22)00320-6.

26. K. H. Wickwire, Optimal isolation policies for deterministic and stochastic epidemics, *Math. Biosci.*, **26**(3-4):325–346, 1975, https://doi.org/10.1016/0025-5564(75)90020-6.

27. X. Xu, P.J. Antsaklis, Optimal control of switched systems via non-linear optimization based on direct differentiations of value functions, *Int. J. Control*, **75**(16-17):1406–1426, 2002, https://doi.org/10.1080/0020717021000023825.

28. Y. Zhou, K. Yang, K. Zhou, Y. Liang, Optimal vaccination policies for an SIR model with limited resources, *Acta Biotheor.*, **62**(2):171–181, 2014, https://doi.org/10.1007/s10441-014-9216-x.

29. F. Zhu, P.J. Antsaklis, Optimal control of switched hybrid systems: A brief survey, *Discrete Event Dyn. Syst.*, **25**(3):345–364, 2015, https://doi.org/10.1007/s10626-014-0187-5.