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A discontinuous nonlinear singular elliptic problem with the fractional ϱ-Laplacian

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Abstract. In this paper, we use the topological degree method, based on the abstract Hammerstein equation, to investigate the existence of weak solutions for a certain class of elliptic Dirichlet boundary value problems. These problems involve the fractional ϱ-Laplacian operator and involve discontinuous nonlinearities in the framework of fractional Sobolev spaces.

Keywords: discontinuous nonlinearities, Hardy potential, fractional ϱ-Laplacian, weak solution, topological degree theory.

1 Introduction and main result

In recent years, significant attention has been directed toward the examination of problems involving discontinuous nonlinearities. In particular, there has been remarkable development in the field of fractional and nonlocal elliptic differential equations. These equations have gained prominence due to their relevance in various domains, including population dynamics, continuum mechanics, game theory, minimal surfaces, phase transition phenomena, image processing, flame propagation, and stratified materials. Comprehensive references on this topic can be found in works such as [9, 10, 16, 19, 21, 24] and the sources cited therein. Notably, the fractional Laplacian can be conceptualized as the infinitesimal generator of a stable Levy process as elaborated in [6, 20, 22, 25]. Consider a real number ϱ with $1 < \varrho < \infty$ and $\mathcal{O}$ as a bounded open set with a smooth boundary in $\mathbb{R}^N$, where $N \geqslant 1$. This study aims to demonstrate the existence of weak nontrivial solutions for the equation defined by

$$\begin{aligned} &(-\Delta)^r_\varrho w + \lambda |w|^{q-2} w + \frac{|w|^{\varrho-2} w}{|z|^{r\varrho}} \in -\big[\underline{\sigma}(z,w), \overline{\sigma}(z,w)\big] \quad \text{in } \mathcal{O}, \\ &w = 0 \quad \text{on } \mathbb{R}^N \setminus \mathcal{O}, \end{aligned} \tag{1}$$

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where $\varrho r < qr < N$ with $0 < r < 1$, $\lambda > 0$, and $(-\Delta)_\varrho^r$ is the fractional ϱ-Laplacian operator defined by

$$(-\Delta)_\varrho^r w(z) = 2 \lim_{\varepsilon \to 0} \int\limits_{B_\varepsilon^c(z)} \frac{|w(z) - w(y)|^{\varrho-2}(w(z) - w(y))}{|z - y|^{N+\varrho r}} \, \mathrm{d}y$$

for $z \in \mathbb{R}^N$, where $B_\varepsilon(z) := \{y \in \mathbb{R}^N \colon |z - y| < \varepsilon\}$.

In addition, considerable research effort has gone into solving the quasilinear problem described by the following system:

$$\begin{aligned} &(-\Delta)_\varrho^r w = f(z, w) \quad \text{in } \mathcal{O}, \\ &w = 0 \quad \text{on } \mathbb{R}^N \setminus \mathcal{O}. \end{aligned} \tag{2}$$

It is worth noting that, in the particular case where $\varrho = 2$, Problem (2) reduces to the fractional Laplacian problem. In the context of discontinuous nonlinearities, Bensid investigated problem (2) and established results concerning the existence and multiplicity of solutions for the following system [8]:

$$\begin{aligned} &(-\Delta)^r w = f(w)\mathfrak{J}(w - \varpi) \quad \text{in } \mathcal{O}, \\ &w = 0 \quad \text{on } \mathbb{R}^N \setminus \mathcal{O}. \end{aligned}$$

Here $\mathfrak{J}$ represents the Heaviside function, f is a given function, and $\varpi > 0$. For cases where $\varrho \neq 2$ and when f is a regular nonlinearity, a substantial body of literature exists on problem (2) with various techniques being employed. Interested readers can refer to the papers [3–5] and the references therein for a comprehensive overview of these approaches.

In the case where the nonlinearity f exhibits discontinuity concerning the variable w, the authors of [1] conducted an investigation into problem (2). To be more specific, the form of f they considered was

$$f(z, w) = m(z) \sum_{i=1}^{n} \mathfrak{J}(w - \varpi_i),$$

where $\mu_i > 0$ were subject to the condition

$$\varpi_1 < \varpi_2 < \cdots < \varpi_n \quad \text{for } n \geqslant 1,$$

and $m \in L^\infty(\mathcal{O})$ exhibited sign-changing behaviour. The authors established both the existence and multiplicity of solutions using the nonsmooth critical point theory.

In [2], Achoura investigates a problem similar to (1) with parameters satisfying $\varrho > 1$, $r \in (0, 1)$ $(N > \varrho r)$, $\lambda > 0$, and f being a Carathéodory function subject to an appropriate growth condition. Notably, when $r = 1$, problems resembling (1) have been extensively investigated in the literature. For further insights, interested readers can explore [7, 14, 17], where diverse methodologies have been employed by the authors to establish the existence of solutions for (1).

In our aim to prove the existence of nontrivial weak solutions, we encounter a significant obstacle arising from the inherent nature of the problem. More specifically, the direct application of topological degree methods becomes impractical due to the discontinuous nature of the nonlinear term σ. To address this discontinuity, we will transform the Dirichlet boundary value problem associated with the fractional ϱ-Laplacian operator and its discontinuous nonlinearities, thereby converting it into a new problem governed by a Hammerstein equation.

In pursuit of this goal, we consistently make an assumption regarding $\sigma : \mathcal{O}\times\mathbb{R} \to \mathbb{R}$, which may exhibit discontinuities. To address these discontinuities, we "fill the discontinuity gaps" in ψ by substituting it with an interval $[\underline{\sigma}(z,w), \overline{\sigma}(z,w)]$, where

$$\underline{\sigma}(z,s) = \liminf_{\eta\to s} \sigma(z,\eta) = \lim_{\delta\to 0^+} \inf_{|\eta-s|<\delta} \sigma(z,\eta),$$
$$\overline{\sigma}(z,s) = \limsup_{\eta\to s} \sigma(z,\eta) = \lim_{\delta\to 0^+} \sup_{|\eta-s|<\delta} \sigma(z,\eta),$$

such that

- (σ_1) The functions $\overline{\sigma}$ and $\underline{\sigma}$ are measurable in superposition, i.e., for any measurable function $w : \mathcal{O} \to \mathbb{R}$, the functions $\overline{\sigma}(\cdot, w(\cdot))$ and $\underline{\sigma}(\cdot, w(\cdot))$ are measurable on $\mathcal{O}$.
- (σ_2) σ satisfying the growth condition $|\sigma(z,s)| \leqslant b(z) + c|s|^{\varrho/\varrho'}$, for a.e. $z \in \mathcal{O}$ and every $s \in \mathbb{R}$, where $b \in L^{\varrho'}(\mathcal{O})$, c is a positive constant.

First, we define the operator $\mathcal{N}$ from $\mathcal{W}_0$ into $2^{\mathcal{W}_0^*}$ as follows:

$$\mathcal{N}w = \biggl\{\varphi \in \mathcal{W}_0^* \biggm| \exists\, g \in L^{\varrho'}(\mathcal{O});\ \underline{\sigma}\bigl(z,w(z)\bigr) \leqslant g(z) \leqslant \overline{\sigma}\bigl(z,w(z)\bigr) \text{ a.e. } z \in \mathcal{O} \\ \text{and } \langle\varphi,\vartheta\rangle = \int_{\mathcal{O}} gv\,\mathrm{d}x \text{ for all } \vartheta \in \mathcal{W}_0\biggr\}.$$

In the sequel, we consider $\mathcal{K} : \mathcal{W}_0 \to \mathcal{W}_0^*$ defined by

$$\langle\mathcal{K}w,\vartheta\rangle = \int_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho-2}(w(z)-w(y))(\vartheta(z)-\vartheta(y))}{|z-y|^{N+s\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int_{\mathcal{O}} \frac{|w(z)|^{\varrho-2}}{|z|^{r\varrho}} w(z)\vartheta(z)\,\mathrm{d}z \tag{3}$$

for any $\vartheta \in \mathcal{W}_0$ with the introduction of the space $\mathcal{W}_0$ in the forthcoming section. Following that, we proceed to define weak solutions for problem (1).

Definition 1. A function $w \in \mathcal{W}_0$ is called a weak solution to problem (1) if there exists an element $\varphi \in \mathcal{N}w$ satisfying

$$\langle\mathcal{K}w,\vartheta\rangle + \lambda\int_{\mathcal{O}} |w|^{q-2}w\vartheta\,\mathrm{d}z + \langle\varphi,\vartheta\rangle = 0 \quad \text{for all } \vartheta \in \mathcal{W}_0.$$

The main result of this paper is summarized in the following theorem.

Theorem 1. *Under assumptions* (σ_1) *and* (σ_2)*, problem* (1) *has a weak solution* w *in* $\mathcal{W}_0$.

2 Preliminaries

Let $\mathcal{O} \subset \mathbb{R}^N$ ($N \geqslant 1$) be a bounded open set with a Lipschitz boundary, and let $\varrho \in \mathbb{R}$ such that $1 < \varrho < \infty$. We begin by selecting the fractional exponent $r \in (0,1)$, and we define the fractional Sobolev space $W^{r,\varrho}(\mathcal{O})$ as follows:

$$W^{r,\varrho}(\mathcal{O}) = \left\{ w \in L^{\varrho}(\mathbb{R}^N)\colon \frac{|w(z) - w(y)|}{|z-y|^{N/\varrho + r}} \in L^{\varrho}\big(\mathbb{R}^{2N}\big) \right\}$$

equipped with the norm

$$\|w\|_{r,\varrho} = \big(\|w\|_{\varrho}^{\varrho} + [w]_{r,\varrho}^{\varrho}\big)^{1/\varrho},$$

where $\|\cdot\|_{\varrho}$ is the norm in $L^{\varrho}(\mathbb{R}^N)$, and $[\cdot]_{r,\varrho}$ is the Gagliardo seminorm defined as

$$[w]_{r,\varrho} = \left(\int\limits_{\mathbb{R}^{2N}} \frac{|w(z) - w(y)|^{\varrho}}{|z-y|^{N + r\varrho}} \,\mathrm{d}z \,\mathrm{d}y \right)^{1/\varrho}.$$

It is worth noting that $W^{r,\varrho}(\mathbb{R}^N)$ is a separable and reflexive Banach space if $1 \leqslant \varrho < \infty$ and $1 < \varrho < \infty$, respectively; see [12].

Now, let us consider problem (1) within the closed linear subspace defined as

$$\mathcal{W}_0 = \big\{ w \in W^{r,\varrho}\big(\mathbb{R}^N\big)\colon\ w = 0 \ \text{ a.e. on } \mathbb{R}^N \setminus \mathcal{O} \big\}.$$

This subspace employs an equivalent norm given by $\|\cdot\| := [\cdot]_{r,\varrho}$; see [13]. It is important to mention that $\mathcal{W}_0$ is a uniformly convex Banach space; see [26, Lemma 2.4].

Let $1 < \varrho r < N$, there exists a positive constant c_H, and we can state the fractional Hardy inequality as follows:

$$\int\limits_{\mathbb{R}^N} \frac{|w(z)|^{\varrho}}{|z|^{r\varrho}} \,\mathrm{d}z \leqslant c_H \int\limits_{\mathbb{R}^{2N}} \frac{|w(z) - w(y)|^{\varrho}}{|z-y|^{N + r\varrho}} \,\mathrm{d}z \,\mathrm{d}y \quad \text{for each } w \in \mathcal{W}_0.$$

For a more comprehensive understanding, we refer to [15].

Moreover, it is well known that the Banach space $(\mathcal{W}_0, \|\cdot\|)$ is uniformly convex, and thus, reflexive. This space is continuously embedded in $L^q(\mathcal{O})$ for all $q \in [1, \varrho^*]$ and compactly embedded in $L^q(\mathcal{O})$ for every $q \in [1, \varrho^*)$, there is a constant c_q, the optimal embedding constant, such that

$$\|w\|_{L^q(\mathcal{O})} \leqslant c_q \|w\| \quad \text{for any } w \in \mathcal{W}_0.$$

Here ϱ^* denotes the fractional critical Sobolev exponent defined as

$$\varrho^* := \begin{cases} \frac{N\varrho}{N - r\varrho} & \text{if } r\varrho < N, \\ +\infty & \text{if } r\varrho \geqslant N. \end{cases}$$

In what follows, we will present the theory of topological degree, an essential tool for establishing our results. To begin, we will introduce several classes of mappings.

Let Γ be a real separable reflexive Banach space with its dual denoted by Γ^*. The symbol $\langle\cdot,\cdot\rangle$ denotes the dual pairing between Γ^* and Γ in this order, and $\rightharpoonup$ denotes weak convergence.

Now, let $\mathcal{A}$ be another real Banach space.

Definition 2. We define a set-valued operator $\mathfrak{B} : \mathcal{O} \subset \Gamma \to 2^{\mathcal{A}}$ as bounded if $\mathfrak{B}$ maps bounded sets to bounded sets.

Definition 3. Consider a nonempty subset $\mathcal{O}$ of Γ, a sequence (w_n) contained in $\mathcal{O}$, and a mapping $\mathfrak{B} : \mathcal{O} \subset \Gamma \to 2^{\Gamma^*} \setminus \emptyset$. Then

(i) $\mathfrak{B}$ is said to have the property of type (S_+) if the following conditions are met:
- If w_n weakly converges to w in Γ and for every sequence (Z_n) in Γ^* such that Z_n belongs to $\mathfrak{B}\varphi_n$ and satisfies $\limsup_{n\to\infty}\langle Z_n,\, w_n - w\rangle \leqslant 0$, then we conclude that w_n converges to w in Γ.

(ii) $\mathfrak{B}$ is called quasimonotone if the following conditions are satisfied:
- If w_n weakly converges to w in Γ and for every sequence (φ_n) in Γ^* such that φ_n belongs to $\mathfrak{B}w_n$, $\liminf_{n\to\infty}\langle \varphi_n,\, w_n - w\rangle \geqslant 0$.

Definition 4. Let $\mathcal{O}$ be a nonempty subset of Γ such that $\mathcal{O}$ is a subset of a larger set $\mathcal{O}_1$, and let (w_n) be a sequence contained within $\mathcal{O}$. Consider a bounded operator $\mathfrak{D} : \mathcal{O}_1 \subset \Gamma \to \Gamma^*$. Then we define the set-valued operator $\mathfrak{B} : \mathcal{O} \subset \Gamma \to 2^{\Gamma} \setminus \emptyset$ to be of type $(S_+)_{\mathfrak{D}}$ if the following conditions hold:

(i) w_n weakly converges to w in Γ, and $\mathfrak{D}w_n$ weakly converges to $\mathcal{A}$ in Γ^*, i.e., $w_n \rightharpoonup w$ in Γ, $\mathfrak{D}w_n \rightharpoonup \mathcal{A}$ in Γ^*.

(ii) For any sequence (Z_n) in Γ with $s_n \in \mathfrak{B}w_n$ such that the limit $\limsup_{n\to\infty}\langle Z_n,\, \mathfrak{D}w_n - \mathcal{A}\rangle \leqslant 0$, we conclude that w_n converges to w in Γ.

We now consider for each $\mathcal{O} \subset D_{\mathfrak{B}}$ and any bounded operator $\mathfrak{D} : \mathcal{O} \to \Gamma^*$, the following sets:

$$\begin{aligned}\mathfrak{B}_1(\mathcal{O}) &:= \big\{\mathfrak{B} : \mathcal{O} \to \Gamma^* \;\big|\; \mathfrak{B} \text{ is demicontinuous, bounded} \\ &\qquad \text{and satifies condition } (S_+)\big\},\\ \mathfrak{B}_{\mathfrak{D}}(\mathcal{O}) &:= \big\{\mathfrak{B} : \mathcal{O} \to 2^{\Gamma} \;\big|\; \mathfrak{B} \text{ is w.u.s.c., locally bounded} \\ &\qquad \text{and satifies condition } (S_+)_{\mathfrak{D}}\big\}.\end{aligned}$$

Lemma 1. (See [18, *Lemma* 1.4].) *Let Θ be a bounded open set in the real reflexive Banach space Γ. Suppose that $\mathfrak{D}$ and that $\mathfrak{S} : \mathcal{D}_{\mathfrak{S}} \subset \Gamma^* \to 2^{\Gamma}$ is locally bounded and weakly upper semicontinuous with $\mathfrak{D}(\overline{\Theta})$ included in $\mathcal{D}_{\mathfrak{S}}$. Then the following results hold*:

(i) *If $\mathfrak{S}$ is quasimonotone, then $\mathcal{I} + \mathfrak{S}\circ\mathfrak{D} \in \mathfrak{B}_{\mathfrak{D}}(\overline{\Theta})$, where $\mathcal{I}$ represents the identity operator.*

(ii) *If $\mathfrak{S}$ is of type (S_+), then $\mathfrak{S}\circ\mathfrak{D} \in \mathfrak{B}_{\mathfrak{D}}(\overline{\Theta})$.*

Remark 1. The operator $\mathfrak{D}$ is an "essential inner map" of $\mathfrak{B}$ if and only if $\mathfrak{D}$ belongs to the set $\mathfrak{B}_1(\overline{\Theta})$.

Definition 5. (See [18].) Suppose $\mathfrak{D} : \overline{\Theta} \subset \Gamma \to \Gamma^*$ is a bounded operator. A homotopy $\mathcal{H} : [0,1] \times \overline{\Theta} \to 2^{\Gamma}$ is considered to be of type $(S_+)_{\mathfrak{D}}$ if, for every sequence $(s_\varepsilon, w_\varepsilon)$ in $[0,1] \times \overline{\Theta}$ and each sequence (a_ε) in Γ with a_ε belonging to $\mathcal{H}(s_\varepsilon, w_\varepsilon)$ such that $w_\varepsilon \rightharpoonup w \in \Gamma$, $s_\varepsilon \to s \in [0,1]$, $\mathfrak{D}w_\varepsilon \rightharpoonup \mathcal{A}$ in Γ^*, and $\limsup_{\varepsilon\to\infty}\langle a_\varepsilon, \mathfrak{D}w_\varepsilon - \mathcal{A}\rangle \leqslant 0$, we have $w_\varepsilon \to w$ in Γ.

Lemma 2. (*See* [18].) *Let Γ be a real reflexive Banach space, and $\Theta \subset \Gamma$ is a bounded open set, $\mathfrak{D} : \overline{\Theta} \to \Gamma^*$ is bounded and continuous. If $\mathfrak{B}, \mathfrak{S}$ are bounded and of class $(S_+)_{\mathfrak{D}}$, then an affine homotopy $\mathcal{H} : [0,1] \times \overline{\Theta} \to 2^{\Gamma}$, giving by*

$$\mathcal{H}(s,w) := (1-s)\mathfrak{B}w + s\mathfrak{S}w \quad \textit{for } (s,w) \in [0,1] \times \overline{\Theta},$$

is of type $(S_+)_{\mathfrak{D}}$.

We introduce the topological degree for the class $\mathfrak{B}_{\mathfrak{D}}(\overline{\Theta})$. For further information, see [18].

Theorem 2. *Let us consider the set $\mathcal{L}$ defined as follows*:

$$\mathcal{L} = \bigl\{(\mathfrak{B}, \Theta, g)\colon\ \Theta \in \mathcal{O},\ \mathfrak{D} \in \mathfrak{B}_1(\overline{\Theta}),\ \mathfrak{B} \in \mathfrak{B}_{\mathfrak{D}}(\overline{\Theta}),\ g \notin \mathfrak{B}(\partial\Theta)\bigr\}.$$

Then there exists a degree function $\delta : \mathcal{L} \to \mathbb{Z}$, which satisfies the following properties:

(i) *Normalization. For all $g \in \Theta$, we have $\delta(I, \Theta, g) = 1$.*
(ii) *Homotopy invariance. If $\mathcal{H} : [0,1] \times \overline{\Theta} \to \Gamma$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to \Gamma$ is a continuous path in Γ such that $g(s) \notin \mathcal{H}(s, \partial\Theta)$ for every $s \in [0,1]$, then the value of $\delta(\mathcal{H}(s,\cdot), \Theta, g(s))$ is constant for all $s \in [0,1]$.*
(iii) *Solution property. If $\delta(\mathfrak{B}, \Theta, g) \neq 0$, then the equation $g \in \mathfrak{B}w$ has a solution in Θ.*

3 Technical lemmas

In this section, we first present several lemmas.

Lemma 3. *Let $w, \vartheta \in W_0$, and there exists a constant $\zeta \geqslant 1$. Then the nonlinear operator $\mathcal{K}$ is well defined and satisfies the following inequations*:

$$\langle \mathcal{K}w, \vartheta\rangle \leqslant \zeta\|w\|^{\varrho-1}\|v\| \quad \textit{and} \quad \|\mathcal{K}w\|_{\mathcal{W}_0^*} \leqslant \zeta\|w\|^{\varrho-1}.$$

Proof. For all $w, \vartheta \in \mathcal{W}_0$, we have

$$\langle \mathcal{K}w, \vartheta\rangle = \int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho-2}(w(z)-w(y))(\vartheta(z)-\vartheta(y))}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho-2}}{|z|^{r\varrho}} w(z)\vartheta(z)\,\mathrm{d}z$$

$$\begin{aligned}
&\leqslant \int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho-2}|w(z)-w(y)||\vartheta(z)-\vartheta(y)|}{|z-y|^{(N+r\varrho)\,(\varrho-1+1)/\varrho}}\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho-2}}{|z|^{r\varrho\,(\varrho-1+1)/\varrho}}\big|w(z)\big|\big|\vartheta(z)\big|\,\mathrm{d}z \\
&\leqslant \int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho-1}}{|z-y|^{(N+r\varrho)\,(\varrho-1)/\varrho}}\frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{\frac{N+r\varrho}{\varrho}}}\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho-1}}{|z|^{r\varrho\,(\varrho-1)/\varrho}}\frac{|\vartheta(z)|}{|z|^{pr/\varrho}}\,\mathrm{d}z.
\end{aligned}$$

Subsequently, applying the Hölder inequality yields the following result:

$$\begin{aligned}
\langle \mathcal{K}w,\vartheta\rangle &\leqslant \Bigg(\int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\Bigg)^{(\varrho-1)/\varrho}\Bigg(\int\limits_{\mathbb{R}^{2N}} \frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\Bigg)^{1/\varrho} \\
&\quad + \Bigg(\int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z\Bigg)^{(\varrho-1)/\varrho}\Bigg(\int\limits_{\mathcal{O}} \frac{|\vartheta(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z\Bigg)^{1/\varrho}.
\end{aligned}$$

For any β belonging to the open interval $(0,1)$ and positive values of a, b, c, and d, the following inequality is employed:

$$a^{\beta}c^{1-\beta} + b^{\beta}d^{1-\beta} \leqslant (a+b)^{\beta}(c+d)^{1-\beta}.$$

Let us set β as $(\varrho-1)/\varrho$ and define the following values:

$$\begin{aligned}
a &= \int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y, &\qquad b &= \int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z \\
c &= \int\limits_{\mathbb{R}^{2N}} \frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y, &\qquad d &= \int\limits_{\mathcal{O}} \frac{|\vartheta(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z.
\end{aligned}$$

From this we can deduce the following inequality:

$$\begin{aligned}
\langle \mathcal{K}w,\vartheta\rangle &\leqslant \Bigg(\int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|w(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z\Bigg)^{(\varrho-1)/\varrho} \\
&\quad \times \Bigg(\int\limits_{\mathbb{R}^{2N}} \frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|v(z)|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z\Bigg)^{1/\varrho}. \qquad (4)
\end{aligned}$$

Then, based on the fractional Hardy inequality (4), we obtain

$$\begin{aligned}\langle\mathcal{K}w,\vartheta\rangle &\leqslant \Bigg(\int\limits_{\mathbb{R}^{2N}}\frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + c_H\int\limits_{\mathbb{R}^{2N}}\frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\Bigg)^{(\varrho-1)/\varrho}\\ &\quad\times\Bigg(\int\limits_{\mathbb{R}^{2N}}\frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + c_H\int\limits_{\mathbb{R}^{2N}}\frac{|\vartheta(z)-\vartheta(y)|}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\Bigg)^{1/\varrho}\\ &\leqslant (c_H+1)\|w\|^{\varrho-1}\|\vartheta\| \leqslant \zeta\|w\|^{\varrho-1}\|\vartheta\| < +\infty.\end{aligned}$$

Furthermore, we have

$$\|\mathcal{K}w\|_{W_0^*} \leqslant \zeta\|w\|^{\varrho-1}. \qquad \square$$

Lemma 4. *For all* $w,\vartheta\in\mathcal{W}_0$ *and a constant* $\zeta\geqslant 1$*, the operator* $\mathcal{K}$ *satisfies the following inequalities*:

$$\langle\mathcal{K}w-\mathcal{K}\vartheta,\,w-\vartheta\rangle \geqslant \zeta\big(\|w\|^{\varrho-1}-\|\vartheta\|^{\varrho-1}\big)\big(\|w\|-\|\vartheta\|\big).$$

Proof. By direct computation, we obtain

$$\begin{aligned}&\langle\mathcal{K}w-\mathcal{K}\vartheta,\,w-\vartheta\rangle\\ &\quad=\langle\mathcal{K}w,\,w-\vartheta\rangle-\langle\mathcal{K}\vartheta,\,w-\vartheta\rangle\\ &\quad=\int\limits_{\mathbb{R}^{2N}}\frac{|w(z)-w(y)|^{\varrho-2}(w(z)-w(y))((w-\vartheta)(z)-(w-\vartheta)(y))}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\\ &\qquad+\int\limits_{\mathcal{O}}\frac{|w(z)|^{\varrho-2}}{|z|^{r\varrho}}w(z)(w-\vartheta)(z)\,\mathrm{d}z+\int\limits_{\mathcal{O}}\frac{|\vartheta(z)|^{\varrho-2}}{|z|^{r\varrho}}\vartheta(z)(w-\vartheta)(z)\,\mathrm{d}z\\ &\qquad-\int\limits_{\mathbb{R}^{2N}}\frac{|\vartheta(z)-\vartheta(y)|^{\varrho-2}(\vartheta(z)-\vartheta(y))((w-\vartheta)(z)-(w-\vartheta)(y))}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\\ &\quad=\int\limits_{\mathbb{R}^{2N}}\frac{|w(z)-w(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y+\int\limits_{\mathcal{O}}\frac{|w|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z+\int\limits_{\mathbb{R}^{2N}}\frac{|\vartheta(z)-\vartheta(y)|^{\varrho}}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\\ &\qquad+\int\limits_{\mathcal{O}}\frac{|\vartheta|^{\varrho}}{|z|^{r\varrho}}\,\mathrm{d}z-\int\limits_{\mathbb{R}^{2N}}\frac{|w(z)-w(y)|^{\varrho-2}(w(z)-w(y))(\vartheta(z)-\vartheta(y))}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\\ &\qquad+\int\limits_{\mathcal{O}}\frac{|w(z)|^{\varrho-2}}{|z|^{r\varrho}}w(z)\vartheta(z)\,\mathrm{d}z\\ &\qquad-\int\limits_{\mathbb{R}^{2N}}\frac{|\vartheta(z)-\vartheta(y)|^{\varrho-2}(\vartheta(z)-\vartheta(y))(w(z)-w(y))}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y\\ &\qquad+\int\limits_{\mathcal{O}}\frac{|\vartheta(z)|^{\varrho-2}}{|z|^{r\varrho}}\vartheta(z)w(z)\,\mathrm{d}z\end{aligned}$$

$$= \int\limits_{\mathbb{R}^{2N}} \frac{|w(z)-w(y)|^\varrho}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|w|^\varrho}{|z|^{r\varrho}}\,\mathrm{d}z + \int\limits_{\mathbb{R}^{2N}} \frac{|\vartheta(z)-\vartheta(y)|^\varrho}{|z-y|^{N+r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|\vartheta|^\varrho}{|z|^{r\varrho}}\,\mathrm{d}z - \langle \mathcal{K}w,\vartheta\rangle - \langle \mathcal{K}\vartheta, w\rangle.$$

Furthermore, using Lemma 3, we find a constant $\zeta \geqslant 1$ such that

$$\begin{aligned}
&\langle \mathcal{K}w - \mathcal{K}\vartheta,\, w-\vartheta\rangle \\
&\quad \geqslant \|w\|^\varrho + \|\vartheta\|^\varrho - \langle \mathcal{K}w,\vartheta\rangle - \langle \mathcal{K}\vartheta, w\rangle \\
&\quad \geqslant \|w\|^\varrho + \|\vartheta\|^\varrho - \zeta\|w\|^{\varrho-1}\|\vartheta\| - \zeta\|\vartheta\|^{\varrho-1}\|w\| \\
&\quad = -\zeta\left(-\frac{1}{\zeta}\|w\|^\varrho - \frac{1}{\zeta}\|\vartheta\|^\varrho + \|w\|^{\varrho-1}\|\vartheta\| + \|\vartheta\|^{\varrho-1}\|w\|\right) \\
&\quad \geqslant -\zeta\bigl(-\|w\|^\varrho - \|\vartheta\|^\varrho + \|w\|^{\varrho-1}\|\vartheta\| + \|\vartheta\|^{\varrho-1}\|w\|\bigr) \\
&\quad \geqslant \zeta\bigl(\|w\|^{\varrho-1} - \|\vartheta\|^{\varrho-1}\bigr)\bigl(\|w\| - \|\vartheta\|\bigr).
\end{aligned}$$

□

Proposition 1. *The nonlinear operator $\mathcal{K}$ have the following properties*:

(i) $\mathcal{K}: \mathcal{W}_0 \to \mathcal{W}_0^*$ *is a bounded, strictly monotone, and continuous operator.*
(ii) $\mathcal{K}$ *is a mapping of type* (S_+).

Proof. (i) According to Lemma 3, there exists a constant $\zeta \geqslant 1$ such that

$$\bigl|\langle \mathcal{K}w,\vartheta\rangle\bigr| \leqslant \zeta\|w\|^{\varrho-1}\|\vartheta\| \quad \text{for all } w,\vartheta \in \mathcal{W}_0.$$

This inequality clearly demonstrates that the operator $\mathcal{K}$ is both continuous and bounded.

In the sequel, by the well-established Simon inequality (refer to [23] for formula (2.2)), which states that for each $\xi,\eta \in \mathbb{R}^N$ and $\varrho > 1$, there exists a positive constant C_ϱ, where

$$\begin{aligned}
&C_\varrho\bigl\langle |\xi|^{\varrho-2}\xi - |\eta|^{\varrho-2}\eta,\, \xi-\eta\bigr\rangle \\
&\quad \geqslant \begin{cases} |\xi-\eta|^\varrho & \text{if } \varrho \geqslant 2, \\ |\xi-\eta|^2(|\xi|^\varrho + |\eta|^\varrho)^{(\varrho-2)/\varrho} & \text{if } 1<\varrho<2. \end{cases}
\end{aligned} \tag{5}$$

Next, applying Lemma 4 and utilizing inequality (5) for any $w,\vartheta \in \mathcal{W}_0$, where $w \neq \vartheta$, we can observe that if $\varrho \geqslant 2$, then

$$C_\varrho\langle \mathcal{K}w - \mathcal{K}\vartheta,\, w-\vartheta\rangle \geqslant \zeta\bigl(\|w\|^{\varrho-1} - \|\vartheta\|^{\varrho-1}\bigr)\bigl(\|w\| - \|\vartheta\|\bigr) \geqslant 0.$$

In the case where $1<\varrho<2$, we have

$$\begin{aligned}
&C_\varrho^{\varrho/2}\bigl[\langle \mathcal{K}w - \mathcal{K}\vartheta,\, w-\vartheta\rangle\bigr]^{\varrho/2}\bigl(\|w\|^\varrho - \|\vartheta\|^\varrho\bigr)^{(2-\varrho)/2} \\
&\quad \geqslant \zeta\bigl(\|w\|^{\varrho-1} - \|\vartheta\|^{\varrho-1}\bigr)\bigl(\|w\| - \|\vartheta\|\bigr),
\end{aligned}$$

which implies

$$C\big[\mathcal{K}w - \mathcal{K}\vartheta,\, w - \vartheta\rangle\big]^{\varrho/2} \geqslant \zeta\big(\|w\|^{\varrho-1} - \|\vartheta\|^{\varrho-1}\big)\big(\|w\| - \|\vartheta\|\big) \geqslant 0,$$

where $C > 0$ is a constant. Consequently, this leads to the conclusion that the operator $\mathcal{K}$ is strict monotonicity.

(ii) As $\mathcal{W}_0$ constitutes a reflexive Banach space, it is isometrically isomorphic to a locally uniformly convex space. Thus, given that we have already demonstrated that weak convergence and norm convergence imply strong convergence, it suffices to establish $\|w_n\| \to \|w\|$.

Moreover, when $w_n \to w$ weakly in $\mathcal{W}_0$,

$$\limsup_{n\to\infty}\big\langle \mathcal{K}w_n - \mathcal{K}w,\, w_n - w\big\rangle \leqslant 0.$$

We can infer that

$$\lim_{n\to+\infty}\langle \mathcal{K}w_n - \mathcal{K}w,\, w_n - w\rangle = \lim_{n\to+\infty}\langle \mathcal{K}w_n,\, w_n - w\rangle - \langle \mathcal{K}w,\, w_n - w\rangle = 0.$$

By means Lemma 4 and the result from (i), we get

$$\langle \mathcal{K}w_n - \mathcal{K}w,\, w_n - w\rangle \geqslant \zeta\big(\|w_n\|^{\varrho-1} - \|w\|^{\varrho-1}\big)\big(\|w_n\| - \|w\|\big) \geqslant 0.$$

This implies that $\|w_n\| \to \|w\|$ as $n \to \infty$, leading us to the conclusion that $w_n \to w$ strongly in $\mathcal{W}_0$ as $n \to \infty$. □

Proposition 2. (*See* [11].) *For any fixed $z \in \mathcal{O}$, the functions $\overline{\sigma}(z,u)$ and $\underline{\sigma}(z,u)$ exhibit upper semicontinuity* (*u.s.c.*) *on $\mathbb{R}^N$.*

Lemma 5. *The operator $\mathcal{J} : \mathcal{W}_0 \to \mathcal{W}_0^*$ defined by*

$$\langle \mathcal{J}w, v\rangle = -\lambda\int_{\mathcal{O}} |w|^{q-2}wv\,\mathrm{d}z \quad \textit{for } w, v \in \mathcal{W}_0$$

is compact.

Proof. Let $\Psi : \mathcal{W}_0 \to L^{q'}(\mathcal{O})$ be the operator defined as

$$\Psi w(z) := -\big|w(z)\big|^{q-2}w(z) \quad \text{for } w \in \mathcal{W}_0 \text{ and } z \in \mathcal{O}.$$

It is evident that Ψ is a continuous operator. Now, we aim to show that Ψ is bounded. For any $w \in \mathcal{W}_0$, we can observe

$$\|\Psi w\|_{q'} \leqslant \lambda\int_{\mathcal{O}} \big||w|^{q-2}w\big|^{q'}\,\mathrm{d}z = \lambda\int_{\mathcal{O}} |w|^{(q-1)q'}\,\mathrm{d}z \leqslant \lambda\int_{\mathcal{O}} |w|^{q}\,\mathrm{d}z.$$

Applying the compact embedding $\mathcal{W}_0 \hookrightarrow L^q(\mathcal{O})$, we deduce % $\|\Psi w\|_{q'} \leqslant C\|w\|_q$. This demonstrates that Ψ is bounded on $\mathcal{W}_0$. Furthermore, considering that the embedding $\mathcal{I} : \mathcal{W}_0 \to L^q(\mathcal{O})$ is compact, it is a known fact that the adjoint operator $\mathcal{I}^* : L^{q'}(\mathcal{O}) \to \mathcal{W}_0^*$ is also compact. Therefore, $\mathcal{J} = \mathcal{I}^* \circ \Psi : \mathcal{W}_0 \to \mathcal{W}_0^*$ is compact. □

Lemma 6. (*See* [28].) *Let $\mathcal{O} \subset \mathbb{R}^N$ $(N \geqslant 1)$ be a bounded open set with a smooth boundary. Under assumptions (σ_1) and (σ_2), the set-valued operator $\mathcal{N}$ defined above is both upper semicontinuous (u.s.c.), bounded, and compact.*

4 Proof of Theorem 1

We study here the nonlinear problem (1) based on the degree theory introduced in the previous section under assumptions (σ_1) and (σ_2).

We consider the set-valued operator $\mathfrak{S}$ defined as $\mathfrak{S} := \mathcal{J} + \mathcal{N}$, where $\mathcal{J}$ and $\mathcal{N}$ are defined above. Thus, $w \in \mathcal{W}_0$ is a weak solution of (1) if and only if

$$\mathcal{K}w \in -\mathfrak{S}w, \tag{6}$$

where $\mathcal{K}$ is given by (3).

By Proposition 1 and the Minty–Browder theorem on monotone operators (see [27, Thm. 26A]), the inverse operator $\mathfrak{D} := \mathcal{K}^{-1}$, defined from $\mathcal{W}_0^*$ to $\mathcal{W}_0$, exists, is bounded, continuous, and of type (S_+). Moreover, by Lemma 5, the operator $\mathfrak{S}$ is bounded, upper semicontinuous, and quasimonotone.

Consequently, (6) can be rewritten equivalently as

$$w = \mathfrak{D}\vartheta \quad \text{and} \quad \vartheta \in -\mathfrak{S} \circ \mathfrak{D}\vartheta. \tag{7}$$

In order to solve Eqs. (7), we first establish the following lemma.

Lemma 7. *The set*

$$\mathcal{E} := \bigl\{\vartheta \in \mathcal{W}_0^*\colon \vartheta \in -s\mathfrak{S} \circ \mathfrak{D}\vartheta \text{ for some } s \in [0,1]\bigr\}$$

is bounded.

Proof. Let $\vartheta \in \mathcal{E}$. Therefore, $\vartheta + ta = 0$ for all $s \in [0,1]$ such that $a \in \mathfrak{S} \circ \mathfrak{D}\vartheta$. Setting $w := \mathfrak{D}\vartheta$, we put $a = \mathcal{J}w + \varphi$, where $\varphi \in \mathcal{N}w$, specifically, $\langle \varphi, w\rangle = \int_{\mathcal{O}} g(z)w(z)\,\mathrm{d}z$ for each $g \in L^{\varrho'}(\mathcal{O})$ satisfying $\underline{\sigma}(z, w(z)) \leqslant g(z) \leqslant \overline{\sigma}(z, w(z))$ for almost all $z \in \mathcal{O}$.

By applying (σ_2), the Young inequality, and the compact embedding $\mathcal{W}_0 \hookrightarrow L^q(\mathcal{O})$, we obtain

$$\begin{aligned}
\|\mathfrak{D}\vartheta\|^\varrho &= \int\limits_{\mathbb{R}^{2N}} \frac{|w(z) - w(y)|^\varrho}{|z - y|^{N + r\varrho}}\,\mathrm{d}z\,\mathrm{d}y + \int\limits_{\mathcal{O}} \frac{|w(z)|^\varrho}{|z|^{r\varrho}}\,\mathrm{d}z \\
&= \langle \vartheta, \mathfrak{D}\vartheta\rangle \leqslant s\bigl|\langle a, \mathfrak{D}\vartheta\rangle\bigr| = s\biggl|\int\limits_{\mathcal{O}} \bigl(\lambda|w|^{q-2}w + s\bigr)w\,\mathrm{d}z\biggr| \\
&\leqslant s\lambda \int\limits_{\mathcal{O}} |w|^q\,\mathrm{d}z + s\int\limits_{\mathcal{O}} |gw|\,\mathrm{d}z \\
&\leqslant s\lambda \int\limits_{\mathcal{O}} |w|^q\,\mathrm{d}z + C_\varrho s\biggl(\int\limits_{\mathcal{O}} |w|^\varrho\,\mathrm{d}z\biggr)^{1/\varrho} + C_{\varrho'}\frac{s}{\alpha}\biggl(\int\limits_{\mathcal{O}} |g|^{\varrho'}\,\mathrm{d}z\biggr)^{1/\varrho'}
\end{aligned}$$

$$\leqslant s\lambda \int\limits_{\mathcal{O}} |w|^q \,\mathrm{d}z + C_p s\biggl(\int\limits_{\mathcal{O}} |w|^\varrho \,\mathrm{d}z\biggr)^{1/\varrho} + 2C'_\varrho s\biggl(\int\limits_{\mathcal{O}} |b|^{\varrho'} \,\mathrm{d}z\biggr)^{1/\varrho'}$$
$$+ 2CC_{\varrho'} s\biggl(\int\limits_{\mathcal{O}} |w|^\varrho \,\mathrm{d}z\biggr)^{1/\varrho'}$$
$$\leqslant \mathrm{Const}\bigl(\|\mathfrak{D}\vartheta\|^q + \|\mathfrak{D}\vartheta\| + \|\mathfrak{D}\vartheta\|^{\varrho-1} + 1\bigr).$$

Therefore, it is clear that the set $\{\mathfrak{D}\vartheta\colon\ \vartheta \in \mathcal{E}\}$ is bounded.

In view of the boundedness of the operator $\mathfrak{S}$ and based on (7), we conclude that the set $\mathcal{E}$ is bounded in $\mathcal{W}_0^*$.

Thanks to Lemma 7, we can determine a positive constant R such that for any $\vartheta \in \mathcal{E}$, it holds that $\|\vartheta\|_{\mathcal{W}_0^*} < R$. This implies that ϑ lies on the boundary of the ball $B_R(0)$ and satisfies $\vartheta \in -s\mathfrak{S} \circ \mathfrak{D}\vartheta$ for every $\vartheta \in \partial B_R(0)$ and each $s \in [0,1]$.

By applying Lemma 1, we have

$$\mathcal{I} + \mathfrak{S} \circ \mathfrak{D} \in \mathfrak{B}_{\mathfrak{D}}\bigl(\overline{B_R(0)}\bigr) \quad \text{and} \quad \mathcal{I} = \mathcal{K} \circ \mathfrak{D} \in \mathfrak{B}_{\mathfrak{D}}\bigl(\overline{B_R(0)}\bigr).$$

Now, we can introduce the affine homotopy $\mathcal{H} : [0,1] \times \overline{B_R(0)} \to 2^{\mathcal{W}_0^*}$ setting by

$$\mathcal{H}(s,\vartheta) := (1-s)\mathcal{I}\vartheta + s(\mathcal{I} + \mathfrak{S} \circ \mathfrak{D})\vartheta \quad \text{for } (s,\vartheta) \in [0,1] \times \overline{B_R(0)}.$$

Using the properties of the degree, as established in Theorem 2, we infer that

$$\delta\bigl(\mathcal{I} + \mathfrak{S} \circ \mathfrak{D},\ B_R(0), 0\bigr) = \delta\bigl(\mathcal{I}, B_R(0), 0\bigr) = 1.$$

Hence, we can find a function $\vartheta \in B_R(0)$ such that $\vartheta \in -\mathfrak{S} \circ \mathfrak{D}\vartheta$, which means that $w = \mathfrak{D}\vartheta$ is a weak solution of (1). This concludes the proof. □

Author contributions. The authors contributed equally to this paper.

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